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# Two-sided Confidence intervals for reliability functions of some members of location-scale family of distributions under complete and censored samples

S. K. Powar
*Department of Statistics.*
*Smt. Kasturbai Walchand College of Arts & Science, Sangli, Maharashtra- 416416, India*
*E-mail: skpkwc@gmail.com*

***Abstract***
*This study focuses on estimating reliability functions for distributions within the location-scale family such as Normal, Exponential, and Extreme value distribution of type I for minimum that are commonly used in fields like physical and engineering sciences. Accurate reliability estimates are critical for applications in quality control, reliability engineering, hydraulic engineering design, and water resources management, where high-confidence interval estimates are often required from observed, sometimes censored, data. These estimates play a key role in making informed engineering decisions, ensuring safety, and optimizing investments. Estimating reliability functions helps assess product lifespan, supports maintenance planning, and addresses various failure patterns, from initial defects to long-term wear. These insights are crucial for minimizing operational disruptions, controlling costs, and informing strategic planning across industries where product dependability is essential, such as manufacturing, healthcare, and technology. Although reliability interval estimation is highly valuable, its development, particularly for confidence interval (CI) in location-scale distributions, has been relatively limited in research. This paper proposes a novel method to construct CIs for reliability functions in the location-scale family where generalized pivotal quantities (GPQs) are available for the parameters. This approach is demonstrated through the construction of CIs for the reliability functions of several distributions, including Weibull, Pareto, Lognormal, Extreme Value Type-I, Exponential, and Normal, applicable to both complete and Type-II right-censored samples. Results from empirical evaluations indicate that this method achieves coverage probabilities close to the nominal level, even with small samples (as few as five observations) and high censoring levels (up to 70%).*
***Keywords:*** *Reliability, Confidence interval, Generalized Variable approach, Censoring, location-scale family of distributions.*

## I. Introduction

Location-scale family distributions, including the Extreme Value Type-I, Exponential, Normal and non-location-scale family distributions, including two-parameter Weibull, Pareto, Gamma, and Lognormal, are extensively used across scientific fields such as biology, environmental and health sciences, physics, and social sciences. These models also play a pivotal role in meteorology, hydrology, and reliability theory, where they are standard tools for analysing time-to-failure data. Studies by researchers such as Grace and Eagleson (1966), Nathan and McMahon (1990), Kulkarni and Powar (2011) and Powar and Kulkarni (2015), among others, highlight the wide applications of these distributions across different domains.

An essential aspect of using these distributions is estimating CIs for reliability functions, which provides valuable insights into product durability and informs strategies for maintenance in various sectors, including quality control, engineering, healthcare, and manufacturing. For example, in reliability assessments for light bulbs modelled by a Weibull distribution, where a scale parameter (2000 hours) and shape parameter (1.5) suggest a wear-out failure mode, key time intervals—early life, useful life, and wear-out—are analysed.

In the early life phase, reliability assessments help identify initial failures, with an expected reliability of 0.9889 at 100 hours, suggesting high functionality in the initial period. During the useful life phase, with reliability around 0.7022 beyond 1000 hours, products maintain steady performance. Finally, as products enter the wear-out period, reliability declines, with an estimated 15.93% of bulbs expected to last beyond 3000 hours.

This type of reliability interval estimation supports effective lifecycle management, maintenance planning, and cost optimization. By providing structured insights across operational phases, CI estimation for reliability functions in location-scale family distributions is invaluable for decision-making across fields where product longevity and reliability are critical.

In reliability studies and life-testing experiments, researchers frequently face challenges in obtaining complete data on failure times for all tested units. For instance, clinical trials may experience participant dropouts due to budget constraints, while in industrial settings, units might be terminated early to save time and reduce costs. Such instances result in what is termed "censored data." The two primary types of censoring encountered are Type-I and Type-II.

Type-I censoring occurs when the experimental duration, *T*, is predetermined, but the number of observed failures can vary. In contrast, Type-II censoring involves a fixed number of failures, *r*, with the experiment's duration allowed to vary until the specified number of failures is reached. The generalized variable (GV) method introduced in this study is tailored for Type-II singly right-censored samples, maintaining the relevance of pivotal quantities for maximum likelihood estimators (MLEs) within this setup.

Distributions in the location-scale family, such as Normal, Exponential, and Extreme value distribution of type I for minimum and distributions in the non-location-scale family, such as the two-parameter Weibull, Pareto, Gamma, and Lognormal distributions, are widely utilized across disciplines to model data in reliability and life-testing studies. However, despite their extensive use, there has been limited focus on constructing CIs for reliability functions, particularly for small sample sizes. This research addresses this gap by proposing a CI estimation method for reliability functions in these families of distributions, aiming to provide an essential tool for accurate reliability assessments across various scientific and engineering applications.

Regulatory standards frequently require accurate reliability estimates for certain distributions at extended time points, even when sample sizes are small to moderate. This presents a particular challenge for distributions within the location-scale and non-location-scale families, such as the Normal, Exponential, Extreme value distribution of type I for minimum, Weibull, Lognormal, and pareto, which are commonly applied in reliability studies. To address this, the objective of this paper is to present a method for constructing CIs that estimate the reliability of these widely used distributions. Our method is developed to achieve coverage probabilities that align closely with nominal values, accommodating both small sample sizes and various data conditions, including censored and uncensored cases, across different values of *t*. This approach provides a practical solution for more accurate reliability estimates across diverse applications.

This study tackles the statistical issue of constructing CIs for reliability functions within widely used location-scale family distributions using the GV approach, originally proposed by Tsui and Weerahandi (1989) and later extended by Weerahandi (1993). The GV approach is instrumental in deriving GPQs, which are key to accurately estimating CIs for various parametric functions. For a more comprehensive understanding and applications of the GV method, readers may refer to Weerahandi's foundational texts (1995, 2004) as well as practical illustrations by Hannig et al. (2006), which highlight its versatility and applicability.

A generalized pivotal quantity (GPQ) is derived uniquely from observed statistics combined with random variables and operates independently of unknown parameters, differing from traditional pivotal quantities. The GV approach has a notable advantage, as it allows for the construction of a GPQ for a function of parameters by using GPQs specifically tailored for each parameter (Krishnamoorthy et al., 2009). This study introduces a GV-based approach to form two-sided CIs for the reliability functions of distributions within the location-scale family, provided GPQs are available for their parameters. The effectiveness of this approach is evaluated through numerical simulations across commonly used distributions, applied to both uncensored and Type-II singly right-censored sample data.

The structure of this paper is organized as follows: Section 2 provides a review of the foundational concepts of GPQs and introduces the proposed methodology for constructing CIs for the reliability functions of location-scale and non-location-scale family distributions. In Section 3, we describe the process of deriving CIs for the reliability functions of some members of these families. Section 4 discusses the estimation of CIs through simulation experiments, with a focus on assessing the coverage probabilities. Section 5 concludes with final observations and recommendations.

## II. Confidence Interval Formulation Using the GPQ Approach:

A GPQ, represented by $G_\theta$ for a parameter $\theta$, is defined by the random variable (RV) $T_\theta(X;x)$, where $X$ is a RV with a distribution influenced by both the target parameter $\theta$ and an additional nuisance parameter $\delta$. The observed value of $X$ is represented as $x$, and $T_\theta(X;x)$ must satisfy two key conditions:

1. When $X = x$, the value $G_\theta = T_\theta(X;x)$ does not depend on the nuisance parameter $\delta$, and often $G_\theta = \theta$ directly.
2. For a given $X = x$, the distribution of $G_\theta = T_\theta(X;x)$ is entirely independent of any unknown parameters.

### 2.1 Estimating Population Reliability: A New Confidence Interval Approach

Consider a random sample $X_1, X_2, \ldots, X_n$ of size *n* taken from a distribution with a probability density function (pdf) $f_X(x;\underline{\theta})$, where $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ represents a vector of unknown parameters. Suppose that each component of $\underline{\theta} \in \Theta \subseteq \Re^k$ has an associated GPQ, denoted by $G_{\underline{\theta}} = \left(G_{\theta_1}, G_{\theta_2}, \ldots, G_{\theta_k}\right)$. Let $R\left(t, \underline{\theta}\right)$ denote the reliability function associated with *X*. Although $R(t, \underline{\theta})$ does not always have an analytical expression, it can be numerically calculated for specified values of *t* and $\underline{\theta}$. We can express a GPQ for $R(t, \underline{\theta})$ as:

$G_{R_t} = R(t, G_{\underline{\theta}})$, t > 0 (1)

where $G_{R_t}$ has a distribution independent of the parameter vector $\underline{\theta}$. Based on this GPQ, a two-sided CI for $R(t, \underline{\theta})$ at a confidence level of (1−λ)×100%, 0 ≤ λ ≤1, can be constructed as follows:

1. For the observed data $\underline{x}$ and the maximum likelihood estimates (or other suitable estimates) $\widehat{\theta_0}$ of $\theta$, repeat the following steps *N* times (e.g., *N*=10,00,000):

I. Calculate the GPQs $G_{\underline{\theta}} = (G_{\theta_1}, G_{\theta_2}, \dots, G_{\theta_k})$ for $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_k)$, possibly using the approach suggested by Iyer and Patterson (2002).

II. Use the expression (1) above to determine $G_{R_t}$.

2. The percentiles (100×λ/2) and 100×(1− λ/2) of the generated *N* values of $G_{R_t}$ define the lower (L) and upper (U) bounds of the two-sided (1−λ)×100% CI for $R(t, \underline{\theta})$, denoted by [L, U]. This interval is referred to as the "Generalized Confidence Interval (GCI)" for $R(t, \underline{\theta})$.

Inference based on GPQs is recognized for providing accurate results; see Roy and Bose (2009) for further details.

## 2.2 Reliability CIs for Monotone Transformed Distributions with GPQs

The following result provides a method for deriving the two-sided CI for the reliability function $R(t, \underline{\theta})$ of the distribution associated with any one-to-one monotonic transformation of the underlying RV.

**Theorem 1**: Let *X* be a continuous RV with pdf $f_X(x; \underline{\theta})$, and reliability function $R(t, \underline{\theta})$, *t* > 0. Suppose that [L, U] is a two-sided GCI for $R(t, \underline{\theta})$. If *Y=h(X)* is a one-to-one monotonic transformation of *X*, and the inverse transformation $X=h^{-1}(Y)$ exists, then the two-sided GCI for the reliability function of the distribution of *Y* is given by:

i. [$h$(L), $h$(U)] when *h'(X)* > 0

ii. [$h$(U), $h$(L)] when *h'(X)* < 0

The proof of the above theorem is direct and follows from standard principles in statistical inference.

## 2.3 Estimating CIs for Reliability Functions in Location-Scale Families

In this section, we utilize the proposed method to derive a two-sided CI for the reliability function $R_{YLS}(t, \mu, \sigma)$ of a RV *Y* that belongs to the location-scale family. The cumulative distribution function (CDF) for *Y* is defined as:

$F_Y(y, \mu, \sigma) = F\left(\frac{y-\mu}{\sigma}\right); \ -\infty < y < \infty, -\infty < \mu < \infty, \sigma > 0$ (2)

where *F* (·) represents the CDF of *Y* when $\mu = 0$ and $\sigma = 1$ and this form is independent of any unknown parameters. The two parameters $(\mu, \sigma)$ are referred to as the location-scale parameters, with $\mu$ representing the location and $\sigma$ representing the scale. When $\sigma = 1$, this forms a subfamily known as the location family, with $\mu$ as the parameter. If $\mu = 0$, it represents the scale family with $\sigma$ as the parameter.

Let $\hat{\mu}$ and $\hat{\sigma}$ denote the MLEs for $\mu$ and $\sigma$, respectively, derived from a complete sample or a Type-II singly right-censored sample where only the smallest *r* observations $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(r)}$ are available. For a complete random sample $Y_1, Y_2, \dots, Y_n$ of size *n*, the pivotal functions $\frac{\hat{\sigma}}{\sigma}$, $\frac{\hat{\mu}-\mu}{\sigma}$, and $\frac{\hat{\mu}-\mu}{\hat{\sigma}}$ are independent of $\mu$ and $\sigma$, as shown by Antle and Bain (1969). This result implies that the ratio $\frac{\hat{\mu}-\mu}{\hat{\sigma}}$ has the same distribution as $\frac{\tilde{\mu}}{\tilde{\sigma}}$ and the ratio $\frac{\hat{\sigma}}{\sigma}$ has the same distribution as $\tilde{\sigma}$ where $\tilde{\mu}$ and $\tilde{\sigma}$ are the MLEs based on a complete random sample with $\mu = 0$ and $\sigma = 1$. A similar result holds for a Type-II singly right-censored sample from the distribution in equation (2), as demonstrated by Lawless (2003, Theorem E2, p. 562).

Using these results, the GPQs for $\mu$ and $\sigma$ are given by:

$G_\mu = \hat{\mu}_0 - \frac{\hat{\mu}-\mu}{\hat{\sigma}}\hat{\sigma}_0 = \hat{\mu}_0 - \frac{\tilde{\mu}}{\tilde{\sigma}}\hat{\sigma}_0$ (3)

$G_\sigma = \frac{\sigma}{\hat{\sigma}}\hat{\sigma}_0 = \frac{\hat{\sigma}_0}{\tilde{\sigma}}$ (4)

where $\hat{\mu}_0$ and $\hat{\sigma}_0$ are the observed values of $\hat{\mu}$ and $\hat{\sigma}$, respectively.

Finally, using equation (1), the GPQ for the reliability function, $R_{YLS}(t, \mu, \sigma)$, of the location-scale family distribution, denoted as $G_{R_{tLS}}$, is given by:

$G_{R_{tLS}} = R_{YLS}(t, G_\mu, G_\sigma)$; $t > 0$ (5)

The following outlines the methodology to compute a two-sided (1− λ)100% GCI for $R_{YLS}(t,\mu,\sigma)$ where $t > 0$, using the algorithm from Section 2:

**Algorithm to obtain two-sided GCI for $R_{YLS}(t,\mu,\sigma)$:**

1. **Sample Generation and Parameter Estimation:**

I. Generate $n$ independent and identically distributed (iid) RVs $y_1, y_2, \dots, y_n$ following the distribution defined in equation (2).

II. Calculate the maximum likelihood estimates $\hat{\mu}_0$ and $\hat{\sigma}_0$ for the location parameter $\mu$ and the scale parameter $\sigma$, respectively.

2. **Monte Carlo Simulation Steps:**

For the estimated $\hat{\mu}_0$ and $\hat{\sigma}_0$, perform the following steps repeatedly for a large number of iterations, say N=10,00,000:

1. Generate a sample of $n$ iid RVs $y_{1_{01}}, y_{2_{01}}, \dots, y_{n_{01}}$ from the distribution in equation (2) with $\mu = 0$ and $\sigma = 1$.
2. Compute the MLEs $\tilde{\mu}$ and $\tilde{\sigma}$ for the generated sample.
3. Determine the GPQs, $G_\mu$ and $G_\sigma$ using equations (3) and (4).
4. Use equation (5) to calculate the GPQ $G_{R_{tLS}}$.

3. **Constructing the GCI:**

The two-sided (1− λ)100% GCI for $R_{YLS}(t,\mu,\sigma)$ is given by:

$\left[G_{R_{tLS}}\left(\frac{\lambda}{2}\right),\ G_{R_{tLS}}\left(1-\frac{\lambda}{2}\right)\ \right]$

where $G_{R_{tLS}}(\lambda)$ is the $(100 \times \lambda)$th percentile of the $G_{R_{tLS}}$ distribution.

For a Type-II singly right-censored sample, the largest $n-r$ observations are excluded. The MLEs (or equivariant estimators) $\tilde{\mu}$ and $\tilde{\sigma}$ are then derived based on the smallest $r$ values $y_{(1)} \le y_{(2)} \le \cdots \le y_{(r)}$. Nkurunziza and Chen (2011) introduced a method to construct GPQs $G_\mu$ and $G_\sigma$ using Pitman estimators, which are also referred to as minimum risk equivariant estimators, for samples from equation (2).

### 2.4 Application to Non-Location-Scale Distributions:

Theorem 1 allows the transformation of a non-location-scale distribution into a location-scale form, facilitating the computation of GCIs for its reliability function. For example:

i. Applying a logarithmic transformation (ln) can convert distributions such as two-parameter Weibull, Pareto, and Log-Normal into Extreme Value Distribution Type I for minimum (EVD-I), Exponential, and Normal distributions, respectively.

ii. These transformed distributions are members of the location-scale family, making them compatible with the described approach for constructing GCIs.

## III. CI estimation of reliability functions of some popular distributions within the location-scale and non-location-scale family:

This section focuses on applying the proposed methodology to commonly used distribution families, such as the two-parameter Weibull, Exponential, Normal, Extreme Value Distribution Type I (EVD-I), Pareto, and Lognormal distributions, under scenarios involving complete samples as well as Type-II singly right-censored samples.

### 2.5 CI for reliability function of Weibull distribution:

The Weibull distribution, characterized by its scale parameter $\alpha$ and shape parameter $\beta$, has a pdf defined as:

$$f_X(x;\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right); x > 0, \alpha > 0, \beta > 0.$$

We denote it as $X \rightarrow$ Weibull ($\alpha$, $\beta$). The reliability function at $t$, for Weibull ($\alpha$, $\beta$) distribution is,

$$R_W(t,\alpha,\beta) = exp\left(-\left(\frac{t}{\alpha}\right)^{\beta}\right); t > 0, \alpha > 0, \beta > 0.$$

For a complete sample, the MLE $\hat{\beta}$ for $\beta$ is the solution to the equation:

$$\frac{1}{\hat{\beta}} - \frac{\sum_{i=1}^{n} x_i^{\hat{\beta}}\, log(x_i)}{\sum_{i=1}^{n} x_i^{\hat{\beta}}} + \frac{1}{n}\sum_{i=1}^{n} log(x_i) = 0 \qquad (6)$$

with $\hat{\alpha} = \left(\sum_{i=1}^{n} x_i^{\hat{\beta}}/n\right)^{1/\hat{\beta}}$.

For a Type-II singly right-censored sample, in which we observe only the smallest $r$ observations, $x_{(1)} \le x_{(2)} \le \cdots \le x_{(r)}$, the MLE for $\beta$ is found by solving the equation:

$\frac{1}{\hat{\beta}} - \frac{\sum_{i=1}^{n} x_{iu}^{\hat{\beta}} \log(x_{iu})}{\sum_{i=1}^{n} x_{iu}^{\hat{\beta}}} + \frac{1}{r}\sum_{i=1}^{r} \log(x_{iu}) = 0$ (7)

and $\hat{\alpha} = \left(\sum_{i=1}^{n} x_{iu}^{\hat{\beta}} /n\right)^{1/\hat{\beta}}$.

Here, $x_{iu} = x_{(i)}$ denotes the observed values in ordered form for $i$ =1, 2,…,$r$ and $x_{iu} = x_{(r)}$ for $i$=$r$+1,…,$n$. The Newton–Raphson method can be applied to iteratively solve the equations (6) and (7), and softwares such as R and MINITAB provides tools to estimate these parameters directly.

**3.1.1 GPQs for parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$, and $\boldsymbol{R_W(t,\alpha,\beta)}$:**

Krishnamoorthy et al. (2009) introduced GPQs for parameters $\alpha$ and $\beta$ as follows. Let $\widehat{\alpha_0}$ and $\widehat{\beta_0}$ denote the observed values of the MLEs $\hat{\alpha}$ and $\hat{\beta}$, respectively. Then, the GPQs for $\alpha$ and $\beta$ can be defined by:

$G_\alpha = \widehat{\alpha_0}\left(\frac{\alpha}{\tilde{\alpha}}\right)^{\tilde{\beta}/\widehat{\beta_0}} = \widehat{\alpha_0}\left(\frac{1}{\tilde{\alpha}}\right)^{\tilde{\beta}/\widehat{\beta_0}}$ (8)

and

$G_\beta = \frac{\beta}{\tilde{\beta}}\,\widehat{\beta_0} = \frac{\widehat{\beta_0}}{\tilde{\beta}}$ (9)

where $\tilde{\alpha}$ and $\tilde{\beta}$ represent the MLEs of $\alpha$ and $\beta$ based on a censored or uncensored sample from a Weibull (1,1) distribution. Using equation (1), the GPQ for $R_W(t,\alpha,\beta)$ can be expressed as:

$$G_{WR_t} = R\left(t, G_\alpha, G_\beta\right) = exp\left(-\left(\frac{t}{G_\alpha}\right)^{G_\beta}\right) = exp\left(-\left(\frac{t(\tilde{\alpha})^{\tilde{\beta}/\widehat{\beta_0}}}{\widehat{\alpha_0}}\right)^{\frac{\widehat{\beta_0}}{\tilde{\beta}}}\right) \quad (10)$$

To compute a two-sided (1− λ)100% GCI for $R_W(t,\alpha,\beta)$ with $t > 0$, based on a complete sample, the following algorithm can be used. This method also applies to Type-II singly right-censored samples, using the relevant MLEs and GPQs.

**Steps of the Algorithm:**

1. Calculate the MLEs $\widehat{\alpha_0}$ and $\widehat{\beta_0}$ for the parameters $\alpha$ and $\beta$ from a sample $x_1, x_2, \dots, x_n$ of size $n$, assuming a Weibull ($\alpha$, $\beta$) distribution.
2. Given the values $\widehat{\alpha_0}$ and $\widehat{\beta_0}$, repeat the following process N times (e.g., N=100,000):

i. Generate $n$ independent random values $x_{1_{11}}, x_{2_{11}}, x_{3_{11}}, \dots, x_{n_{11}}$ from a Weibull(1,1) distribution, then estimate $\tilde{\alpha}$ and $\tilde{\beta}$, the MLEs for $\alpha$ and $\beta$ from this generated data.

ii. Use Equations (8) and (9) to compute the GPQs, $G_\alpha$ and $G_\beta$.

iii. Use Equation (10) to determine $G_{WR_t}$, the GPQ for $R_W(t,\alpha,\beta)$.

The (1− λ)×100% GCI for $R_W(t,\alpha,\beta)$ with $t > 0$ can be expressed as follows:

$\left[G_{WR_t;\lambda/2},\ G_{WR_t;1-\lambda/2}\right]$ (11)

where $G_{WR_t;\lambda}$ represents the (100× λ)th percentile of $G_{WR_t}$.

**2.6 CI for reliability function of Exponential distribution:**

We now examine a two-parameter Exponential distribution, represented as $E(\mu_1,\sigma_1)$, with its pdf defined as:

$$f_X(x,\mu_1,\sigma_1) = \frac{1}{\sigma_1} exp\left(-\frac{x-\mu_1}{\sigma_1}\right); x > \mu_1, -\infty < \mu_1 < \infty, \sigma_1 > 0,$$

where $\mu_1$ is the location parameter and $\sigma_1$ is the scale parameter. The reliability function at $t$, for $E(\mu_1,\sigma_1)$ distribution is given by,

$$R_E(t,\mu_1,\sigma_1) = exp\left(-\frac{t-\mu_1}{\sigma_1}\right); t > \mu_1$$

Consider a random sample $X_1, X_2, \dots, X_n$ of size $n$ drawn from the $E(\mu_1,\sigma_1)$ distribution. The MLEs for the location parameter $\mu_1$ and the scale parameter $\sigma_1$, denoted by $\hat{\mu}_1$ and $\hat{\sigma}_1$, are given by:

$$\hat{\mu}_1 = X_{(1)} = min(X_1, X_2, \dots, X_n)$$

$$\hat{\sigma}_1 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - X_{(1)}\right)$$

The distributions of $\hat{\mu}_1$ and $\hat{\sigma}_1$ are independent. Specifically, $\frac{2n(\hat{\mu}_1-\mu_1)}{\sigma_1} \sim \chi^2_{(2)}$ and $\frac{2n\hat{\sigma}_1}{\sigma_1} \sim \chi^2_{(2n-2)}$, where $\chi^2_{(\vartheta)}$ represents the chi-square distribution with $\vartheta$ degrees of freedom. Let $\hat{\mu}_{10}$ and $\hat{\sigma}_{10}$ denote the observed values of $\hat{\mu}_1$ and $\hat{\sigma}_1$, respectively. Using these, the GPQs for $\mu_1$ and $\sigma_1$ are expressed as follows:

$G_{\mu_1} = \hat{\mu}_{10} - \frac{2n(\hat{\mu}_1 - \mu_1)}{\sigma_1} \frac{\hat{\sigma}_{10}}{\frac{2n\hat{\sigma}_1}{\sigma_1}} = \hat{\mu}_{10} - \frac{\chi^2_{(2)}}{\chi^2_{(2n-2)}} \hat{\sigma}_{10}$ (12)

$G_{\sigma_1} = \frac{2n\hat{\sigma}_{10}}{\frac{2n\hat{\sigma}_1}{\sigma_1}} = \frac{2n\hat{\sigma}_{10}}{\chi^2_{(2n-2)}}$ (13)

In the case of progressive Type-II censoring, MLEs for $\mu_1$ and $\sigma_1$ under a Type-II singly right-censored sample $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ are given as:

$$\hat{\mu}_{1c} = X_{(1)}$$

$$\hat{\sigma}_{1c} = \frac{\sum_{i=1}^{r-1} X_{(i)} + (n - r + 1)X_{(r)} - nX_{(1)}}{r}$$

It has been established that $\hat{\mu}_{1c}$ and $\hat{\sigma}_{1c}$ are independent, with $\frac{2n(\hat{\mu}_{1c} - \mu_1)}{\sigma_1} \sim \chi^2_{(2)}$ and $\frac{2r\hat{\sigma}_{1c}}{\sigma_1} \sim \chi^2_{(2r-2)}$. Let $\hat{\mu}_{1c0}$ and $\hat{\sigma}_{1c0}$ denote the observed values of these estimators. The GPQs for $\mu_1$ and $\sigma_1$, based on this censored sample, can then be represented as:

$$G_{\mu_1} = \hat{\mu}_{1c0} - \frac{2nr(\hat{\mu}_{1c} - \mu_1)}{n\sigma_1} \frac{\hat{\sigma}_{1c0}}{\frac{2r\hat{\sigma}_{1c}}{\sigma_1}} = \hat{\mu}_{1c0} - \frac{r\chi^2_{(2)}}{n\chi^2_{(2r-2)}} \hat{\sigma}_{1c0}$$

$$G_{\sigma_1} = \frac{2r\hat{\sigma}_{1c0}}{\frac{2r\hat{\sigma}_{1c}}{\sigma_1}} = \frac{2r\hat{\sigma}_{1c0}}{\chi^2_{(2r-2)}}$$

Using Equation (1), the GPQ for the reliability function $R_E(t, \mu_1, \sigma_1) = exp\left(-\frac{t-\mu_1}{\sigma_1}\right); t > \mu_1$ of the $E(\mu_1, \sigma_1)$ distribution is defined as:

$G_{ER_t} = R_E(t, G_{\mu_1}, G_{\sigma_1}) = exp\left(-\left(\frac{t - G_{\mu_1}}{G_{\sigma_1}}\right)\right)$ (14)

To calculate a two-sided (1−λ)100% GCI for $R_E(t, \mu_1, \sigma_1)$ for $t > \mu_1$, the following procedure is outlined. This method is also applicable to Type-II singly right-censored samples by using the appropriate MLEs and GPQs for $\mu_1$ and $\sigma_1$.

**Algorithm:**

1. **Generate a Sample**: Simulate *n* independent RVs $x_1, x_2, \ldots, x_n$ from the $E(\mu_1, \sigma_1)$ distribution, and compute the MLEs $\hat{\mu}_{10}$ and $\hat{\sigma}_{10}$.
2. **Perform Simulations**: Using $\hat{\mu}_{10}$ and $\hat{\sigma}_{10}$, repeat the following steps N times (e.g., N=10,00,000):

a. Generate a random value from the $\chi^2_{(2)}$ distribution.

b. Generate a random value from the $\chi^2_{(2n-2)}$ distribution.

c. Calculate the GPQs $G_{\mu_1}$ and $G_{\sigma_1}$ using the relevant equations.

d. Compute the GPQ $G_{ER_t}$ using Equation (14).

3. **Construct the GCI**: The (1− λ)100% GCI for $R_E(t, \mu_1, \sigma_1)$ is given by:

$$\left[G_{ER_t;\lambda/2}, \ G_{ER_t;1-\lambda/2}\right] \quad (15)$$

where $G_{ER_t;\lambda}$ represents the (100×λ)th percentile of the simulated GPQ $G_{ER_t}$.

## 2.7 CI for reliability function of Normal distribution:

The pdf of a Normal distribution, denoted as $N(\mu_2, \sigma_2)$, with mean (location parameter) $\mu_2$ and standard deviation (scale parameter) $\sigma_2$, is given by:

$$f_X(x, \mu_2, \sigma_2) = \frac{1}{\sigma_2\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{x - \mu_2}{\sigma_2}\right)^2\right); -\infty < x < \infty, -\infty < \mu_2 < \infty, \sigma_2 > 0$$

Let $X_1, X_2, \ldots, X_n$ be a random sample of size *n* drawn from $N(\mu_2, \sigma_2)$. The MLE for $\mu_2$ is $\hat{\mu}_2 = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. The MLE for $\sigma_2^2$ is $\hat{\sigma}_2^2 = \frac{(n-1)S^2}{n}$, where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Now define $V_2 = (n - 1)S^2$, $U_2 = V_2/\sigma_2^2$, and $Z = \frac{\bar{X} - \mu_2}{\sigma_2/\sqrt{n}}$. Let $\bar{x}$ and $v_2$ represent the observed values of $\bar{\bar{X}}$ and $V_2$, respectively. It is established that $U_2$

follows a chi-square distribution with $n-1$ degrees of freedom $(\chi^2_{n-1})$, and $Z$ follows a Standard Normal Distribution $N(0,1)$. Moreover, $U_2$ and $Z$ are independent. The GPQ for $\mu_2$ is expressed as:

$$G_{\mu_2} = \bar{x} - \frac{Z}{\sqrt{U_2}}\frac{\sqrt{v_2}}{\sqrt{n}} \quad (16)$$

Here, $Z$ and $U_2$ are RVs simulated from their respective distributions. The GPQ for $\sigma_2$ is given by:

$$G_{\sigma_2} = \frac{\sigma_2^2}{V_2} v_2 \quad (17)$$

For a Type-II singly right-censored sample $X_{(1)} \le X_{(2)} \le \cdots \le X_{(r)}$, the MLEs $\hat{\mu}_{2c}$ and $\hat{\sigma}_{2c}$ for the parameters $\mu_2$ and $\sigma_2$ are obtained by solving the following equations numerically:

$$\sum_{i=1}^{r} D_{(i)} + (n-r)\frac{\phi(D_{(r)})}{\bar{\Phi}(D_{(r)})} = 0 \quad (18)$$

$$-r + \sum_{i=1}^{r} D_{(i)}^2 + (n-r)D_{(r)}\frac{\phi(D_{(r)})}{\bar{\Phi}(D_{(r)})} = 0 \quad (19)$$

where $D_{(i)} = \frac{X_{(i)}-\mu_2}{\sigma_2}$ for $i$=1,2,…,$r$, $\phi(.)$ is the pdf of the standard normal distribution $N$(0,1), and $\Phi(.)$ is its CDF. The survival function is represented as $\bar{\Phi}(.) = 1 - \Phi(.)$.

Let $\hat{\mu}_{2c0}$ and $\hat{\sigma}_{2c0}$ denote the observed MLEs obtained from the censored sample. The GPQs for $\mu_2$ and $\sigma_2$ are then defined as:

$$G_{\mu_2} = \hat{\mu}_{2c0} - \frac{\tilde{\mu}_{2c}}{\tilde{\sigma}_{2c}}\hat{\sigma}_{2c0}$$

$$G_{\sigma_2} = \frac{\hat{\sigma}_{2c0}}{\tilde{\sigma}_{2c}}$$

where $\tilde{\mu}_{2c}$ and $\tilde{\sigma}_{2c}$ are the MLEs obtained from a Type-II singly right-censored sample from $N$(0,1).

Using these, the GPQ for the reliability function of $N(\mu_2, \sigma_2)$, expressed as $R_N(t, \mu_2, \sigma_2) = 1 - \Phi\left(\frac{t-\mu_2}{\sigma_2}\right); t > 0$, is:

$$G_{NR_t} = R_N\left(t, G_{\mu_2}, G_{\sigma_2}\right) = 1 - \Phi\left(\frac{t-G_{\mu_2}}{G_{\sigma_2}}\right),\ t > 0 \quad (20)$$

The procedure to compute a two-sided (1− λ)×100% GCI for $R_N(t, \mu_2, \sigma_2)$ under the censored sample follows the same algorithm described in section 2.2 used for the complete sample case. The resulting CI is:

$$\left[G_{NR_t;\lambda/2},\ G_{NR_t;1-\lambda/2}\right] \quad (21)$$

where $G_{NR_t;\lambda}$ denotes the (100×λ)th percentile of the distribution of $G_{NR_t}$.

**2.8 CIs for reliability functions of EVD-I, Pareto and Lognormal distributions:**

Using one-to-one monotonic transformations, the Weibull, Exponential, and Normal distributions are mapped to the EVD-I, Pareto, and Lognormal distributions, respectively. Based on Theorem 1, two-sided GCIs are derived for the reliability functions of these transformed distributions. The transformation methods and the resulting GCIs are summarized in Table 1.

**Table 1:** GCIs for reliability functions of EVD-I, Pareto, and Log-Normal Models

| Original distribution, notation | Transformed variable | Transformed distribution | Transformed GCIs (equation number) |
|---|---|---|---|
| Weibull $X \rightarrow$ Weibull $(\alpha, \beta)$ | log($X$) | EVD-I | $\left[log(G_{WR_t;\lambda/2}),\ \log(G_{WR_t;1-\lambda/2})\right]$ (22) |
| Exponential $X \rightarrow E(\mu_1, \sigma_1)$ | exp($X$) | Pareto | $\left[exp(G_{ER_t;\lambda/2}),\ \exp(G_{ER_t;1-\lambda/2})\right]$ (23) |
| Normal $X \rightarrow N(\mu_2, \sigma_2)$ | exp($X$) | Lognormal | $\left[\exp(G_{NR_t;\lambda/2}),\ exp(G_{NR_t;1-\lambda/2})\right]$ (24) |

## IV. Evaluations of the CIs:

This empirical study aimed to evaluate the performance of the proposed GCIs. For a significance level of λ=0.05, sample sizes n=5,15,25,50 and considering values of $t$ such that reliability function takes the values 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99 and the parameter configurations detailed in Table 2, a total of 1,00,000 samples were simulated from Weibull, Exponential, Normal, EVD-I, Pareto, and Lognormal distributions.

For each of the 1,00,000 samples, the lower and upper bounds (Li, Ui), where i=1, 2,…,1,00,000 of the two-sided GCIs were determined using Eqs. (11), (15), and (21) – (24). The coverage probability, defined as the proportion of intervals that included the true value of the reliability function, was calculated for each of these GCIs. The study's goal was to assess the performance of the GCI estimators for different sample sizes ($n$) and values of $t$.

Boxplots representing the percentage coverage probabilities for all parameter combinations from Table 2, corresponding to the above mentioned six proposed GCIs, are displayed in Figures 1 and 2 for complete samples. The GV method produced coverage probabilities that were consistently close to the nominal level and exhibited minimal variability. Therefore, the GV method is precise, as corroborated by the findings of Roy and Bose (2009).

Based on the findings, the proposed GV method appears to be the only exact approach currently available and is strongly recommended for practical use.

In the case of Type II censored samples, the proportions of censored observations, represented as $PC = Pr\left(X \geq X_{(r)}\right)$, are considered as $PC = 0.3, 0.5, 0.7$ based on the parametric combinations detailed earlier. For brevity of the manuscript, graphical representations of the results for $PC = 0.3, 0.7$ and $n$ =5, 25, 50 are provided in Figures 3 to 5.

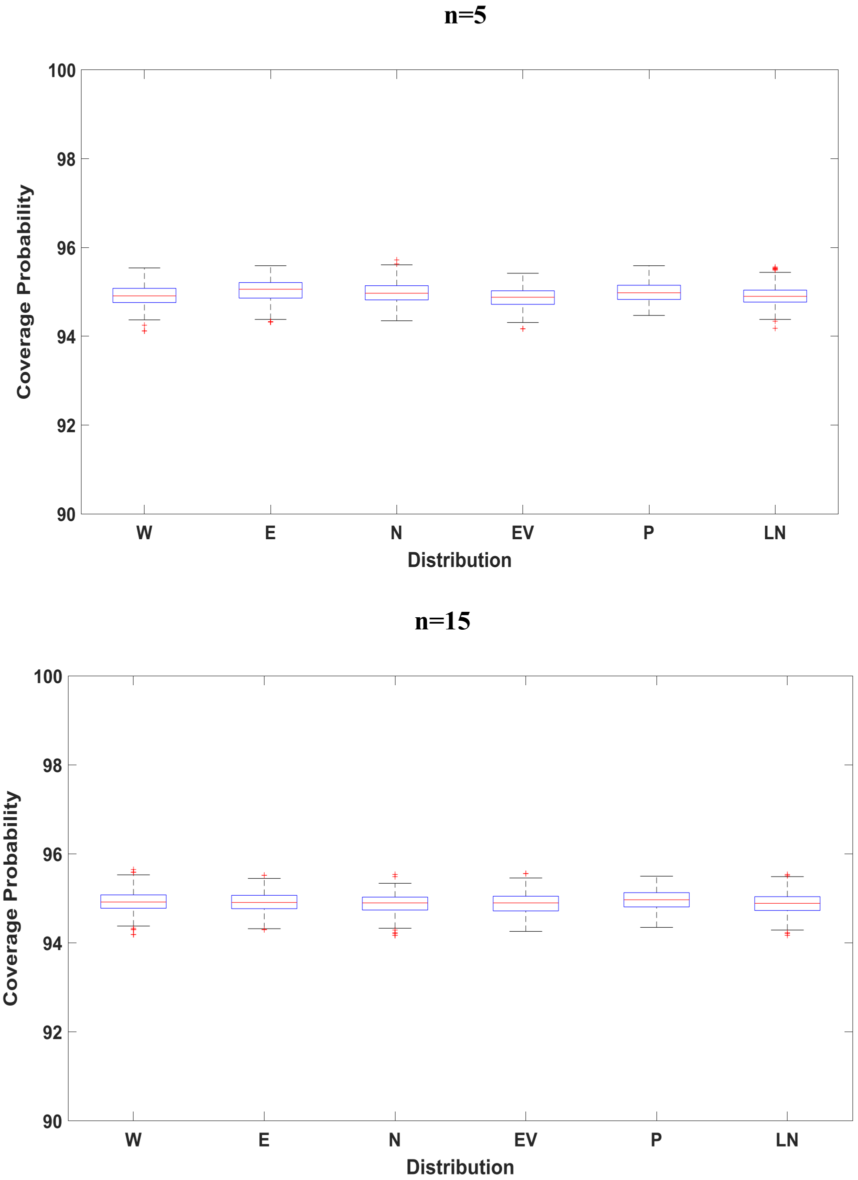
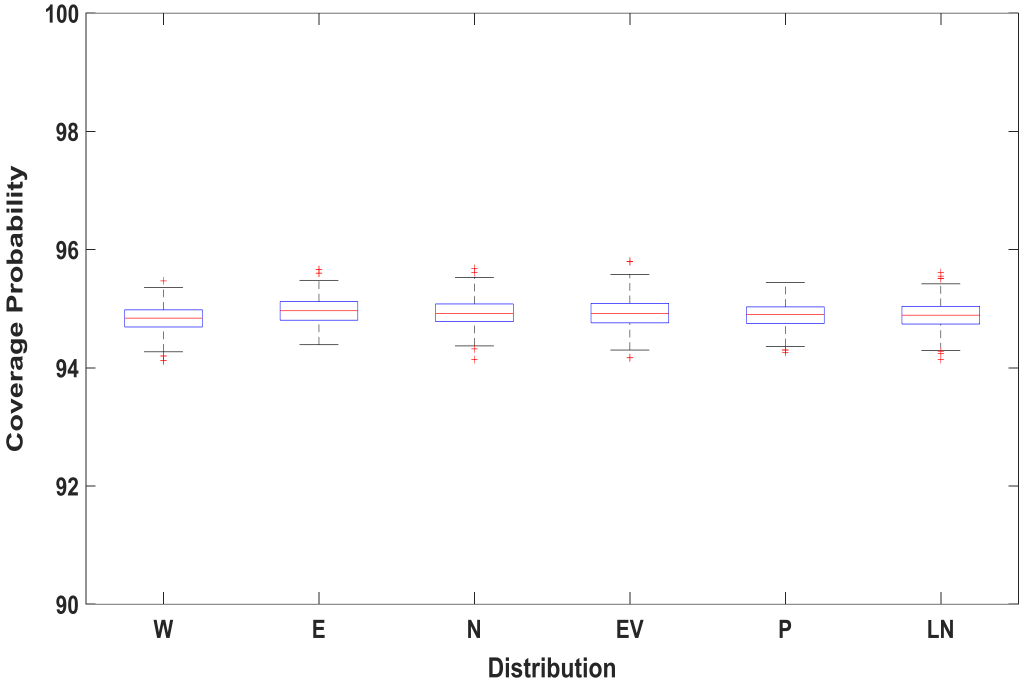



**Fig. 1** Box plots of expected coverage probabilities (in percentage) for 95 % GCIs of reliability functions for the Weibull (W), Exponential (E), Normal (N), EVD-I (EV), Pareto (P) and Lognormal (L) distributions, for sample size n=5 and 15 over the range of the parameter configurations detailed in Table 2 and values of $t$ such that reliability function takes the values 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99.

**n=25**




**n=50**

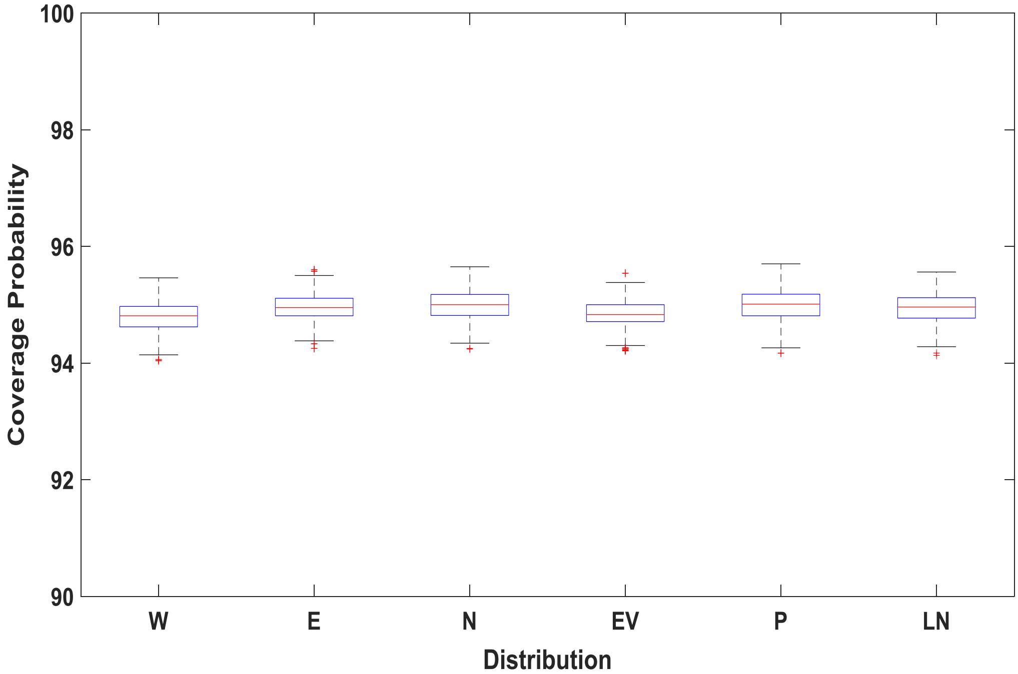
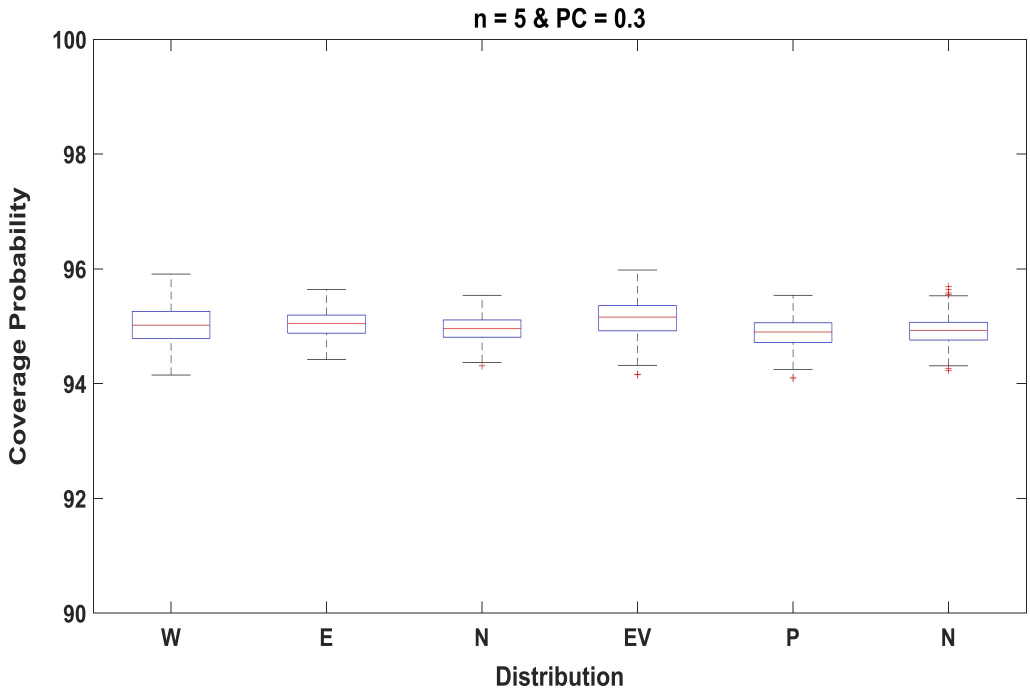



**Fig. 2** Box plots of expected coverage probabilities (in percentage) for 95 % GCIs of reliability functions for the Weibull (W), Exponential (E), Normal (N), EVD-I (EV), Pareto (P) and Lognormal (L) distributions, for sample size n=25 and 50 over the range of the parameter configurations detailed in Table 2 and values of *t* such that reliability function takes the values 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99.



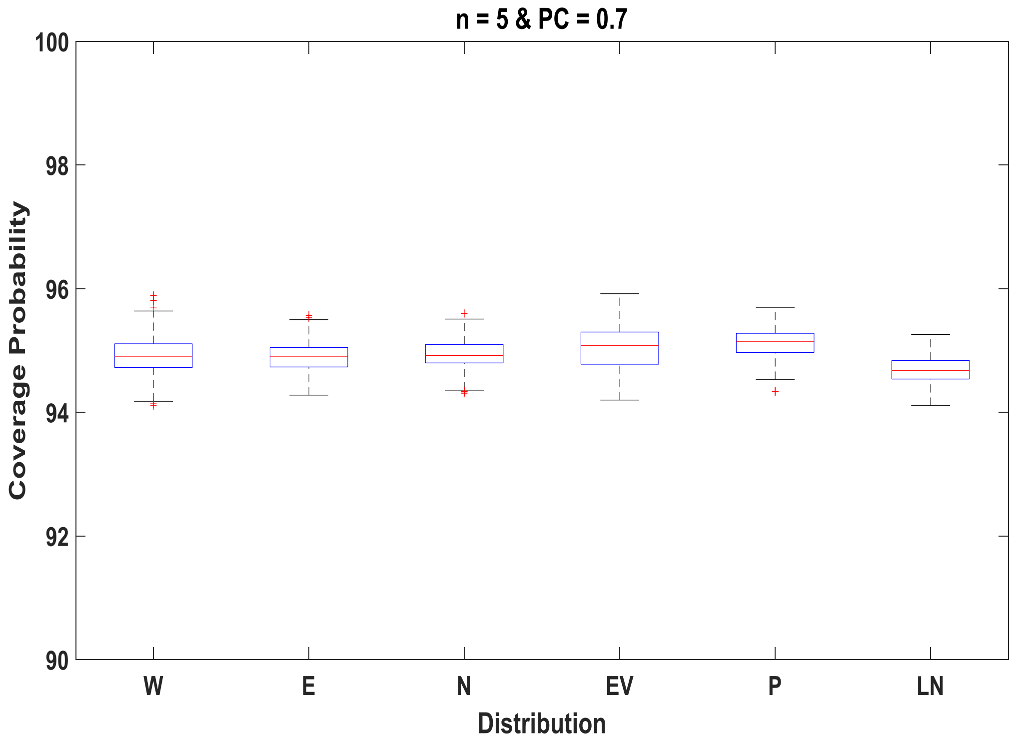




**Fig. 3** Box plots of expected coverage probabilities (in percentage) for 95 % GCIs of reliability functions for the Weibull (W), Exponential (E), Normal (N), EVD-I (EV), Pareto (P) and Lognormal (L) distributions, for Type-II right censored samples of size n=5 with proportion of censoring PC = 0.3 and 0.7 over the range of the parameter configurations detailed in Table 2 and values of *t* such that reliability function takes the values 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99.

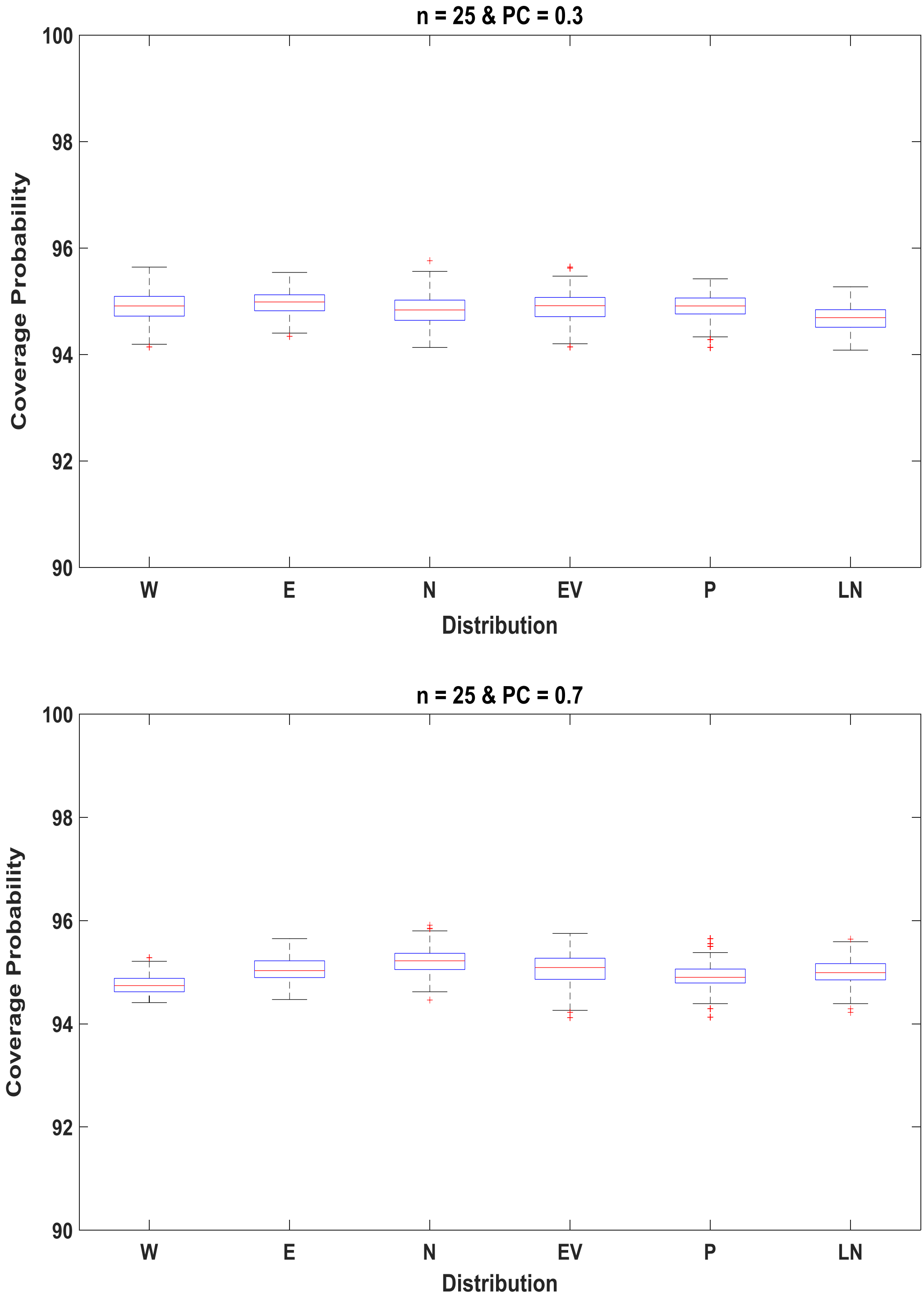



**Fig. 4** Box plots of expected coverage probabilities (in percentage) for 95 % GCIs of reliability functions for the Weibull (W), Exponential (E), Normal (N), EVD-I (EV), Pareto (P) and Lognormal (L) distributions, for Type-II right censored samples of size n=25 with proportion of censoring PC = 0.3 and 0.7 over the range of the parameter configurations detailed in Table 2 and values of *t* such that reliability function takes the values 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99.

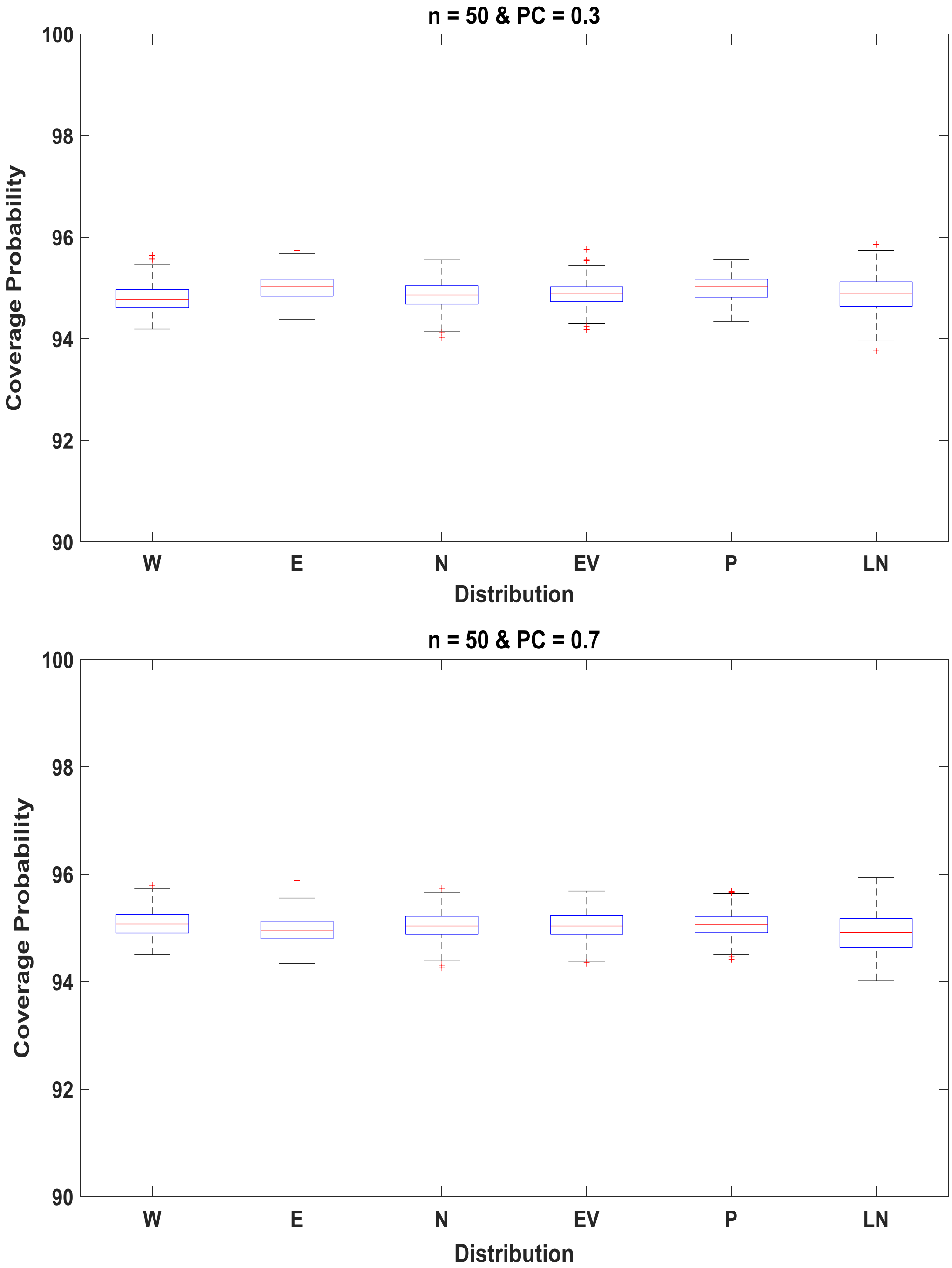



**Fig. 5** Box plots of expected coverage probabilities (in percentage) for 95 % GCIs of reliability functions for the Weibull (W), Exponential (E), Normal (N), EVD-I (EV), Pareto (P) and Lognormal (L) distributions, for Type-II right censored samples of size n=50 with proportion of censoring PC = 0.3 and 0.7 over the range of the parameter configurations detailed in Table 2 and values of *t* such that reliability function takes the values 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99.

All the figures demonstrate that the proposed method maintains coverage probabilities close to 0.95, even for small uncensored sample sizes (as low as 5) and for censored samples with up to 70% censored observations.

**Table 2** Parametric combinations chosen for various distributions

| Distribution | Parameters chosen |
|---|---|
| Weibull | $\alpha$ = 0.5,1,2,…,15.<br>$\beta$= 0.3,0.5,1,2,…,10. |
| Exponential | $\mu_1$= -3,-2,…,3,4,5.<br>$\sigma_1$= 0.5,1,3,5,10,11,…,15. |
| Normal | $\mu_2$= -5,-4,…,5,6,…,10.<br>$\sigma_2$= 0.5,1,3,5,7,10,11,…,15. |
| *EVD-I*<br>$f_X(x;\alpha^*,\beta^*) = \frac{1}{\beta^*}\, exp\left(\left(\frac{x-\alpha^*}{\beta^*}\right) - exp\left(\frac{x-\alpha^*}{\beta^*}\right)\right);$<br>$-\infty < (x,\alpha^*) < \infty, \beta^* > 0.$ | $\alpha^*$= -7,-5,-1,1,5,9,10.<br>$\beta^*$= 0.3,0.5,1,3,5,7,10. |
| Pareto<br>$f_X(x,\mu_1^*,\sigma_1^*) = \frac{\sigma_1^*}{\mu_1^*}\left(\frac{x}{\mu_1^*}\right)^{-\sigma_1^*-1} \quad ; x > \mu_1^*, \mu_1^* > 0, \sigma_1^* > 0.$ | $\mu_1^*$= 0.5,1,2,…,15.<br>$\sigma_1^*$= 1,2,…,15. |
| Lognormal<br>$f_X(x,\mu_2^*,\sigma_2^*) = \frac{1}{x\sigma_2^*\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{\log(x)-\mu_2^*}{\sigma_2^*}\right)^2\right);$<br>$-\infty < x < \infty, -\infty < \mu_2^* < \infty, \sigma_2^* > 0.$ | $\mu_2^*$= -5,-4,…,10.<br>$\sigma_2^*$= 0.5,1,2,…,15. |

## V. Overall conclusions

This research presents a method for constructing CIs for the reliability functions of distributions with GPQs for their parameters. The method is demonstrated for the location-scale family of distributions under complete and Type-II censored sampling. It is simple to apply and delivers coverage probabilities that closely match the nominal level, even for small uncensored samples (as few as 5) and for Type-II right-censored samples with censored proportions up to 70%. The results from simulation studies validate the effectiveness of this approach.

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